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Linear invariance of locally biholomorphic mappings in the unit ball of a JB^* -triple

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ABSTRACT

We study the notion of linear invariance on the unit ball of a JB^* -triple X , and we obtain some connection between the norm-order of a linear invariant family and the starlikeness of order $1/2$. Also, we give some result concerning the radius of univalence of some linear invariant families. Finally, if the dimension of X is finite and if the norm-order of a linear invariant family is finite, then we prove the normality of the linear invariant family and we also obtain upper bounds on the distortion and the growth of mappings in a linear invariant family with specified norm-order. In particular, our results are valid for the classical Cartan domains and the unit polydisc.

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1. Introduction

Linear invariance, introduced by Pommerenke [30,31] has been a powerful tool in extending many ideas of univalent function theory to the study of locally univalent functions on the unit disc. Recently Barnard, FitzGerald and Gong [1], Pfaltzgraff [26], Pfaltzgraff and Suffridge [27–29], Hamada and Kohr [12,13,15,16] and Graham, Hamada, Kohr and Suffridge [5] studied the linear invariant families in several complex variables. Barnard, FitzGerald and Gong [1] studied a linear invariant family on the Euclidean unit ball in \mathbb{C}^2 and Pfaltzgraff [26], Pfaltzgraff and Suffridge [27,28], Godula, Liczberski and Starkov [7], Liczberski and Starkov [24], studied linear invariant families on the Euclidean unit ball in \mathbb{C}^n by using the (trace) order of a linear invariant family.

On the other hand, several interesting results, concerning the norm-order of a linear invariant family and some connections with starlikeness, convexity and other geometric properties of holomorphic mappings in \mathbb{C}^n , were recently obtained by Pfaltzgraff and Suffridge [29]. Also they showed a number of growth, covering and distortion results for mappings that belong to a linear invariant family on the Euclidean unit ball in \mathbb{C}^n . Hamada and Kohr generalized the results in [29] to the unit ball in a complex Hilbert space in [12] and to the unit polydisc in [13]. For linear invariant families in several complex variables, see also the books [3,6] and the references therein.

In this paper we continue the study of linear invariance on the unit ball B of a JB^* -triple. All four types of classical Cartan domains and their infinite dimensional analogues are the open unit balls of JB^* -triples, and the same holds for any finite or infinite product of these domains ([18], see also [11,20]). Thus the unit balls of JB^* -triples are natural generalizations of the unit disc in \mathbb{C} and we have a setting in which a large number of bounded symmetric homogeneous domains may be studied simultaneously. We obtain some connection between the norm-order of a linear invariant family and the starlikeness of order $1/2$. Also, we give some result concerning the radius of univalence of some linear invariant families. Finally, when

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the dimension of X is finite, and the norm-order of a linear invariant family is finite, we will prove the normality of the linear invariant family and we also obtain upper bounds on the distortion and the growth of mappings in a linear invariant family with specified norm-order.

2. Preliminaries

Let B be the unit ball in a complex Banach space X . Let Y be a complex Banach space. A holomorphic mapping $f : B \rightarrow Y$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in B$. A holomorphic mapping $f : B \rightarrow Y$ is said to be biholomorphic if $f(B)$ is a domain in Y , f^{-1} exists and holomorphic on $f(B)$. A biholomorphic mapping $f : B \rightarrow Y$ is said to be convex if $f(B)$ is a convex domain. Let X^* be the dual space of X . For each $x \in X \setminus \{0\}$, we define

$$T(x) = \{x^* \in X^* : \|x^*\| = 1, x^*(x) = \|x\|\}.$$

By the Hahn–Banach theorem, $T(x)$ is nonempty. Let $f : B \rightarrow X$ be a locally biholomorphic mapping. Let $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$. We say that f is a starlike mapping of order α if

$$\left| \frac{1}{\|x\|} x^*([Df(x)]^{-1} f(x)) - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}$$

for $x \in B \setminus \{0\}$, $x^* \in T(x)$.

Let $L(X, Y)$ denote the set of continuous linear operators from X into Y . Let I_X be the identity in $L(X, X)$.

Let $\mathcal{LS}(B)$ denote the family of locally biholomorphic mappings from B to X , normalized by $f(0) = 0$ and $Df(0) = I_X$.

We recall that a JB*-triple is a complex Banach space X together with a continuous mapping (called Jordan triple product)

$$X \times X \times X \rightarrow X, \quad (x, y, z) \mapsto \{x, y, z\}$$

such that for all elements in X the following conditions (J₁)–(J₄) hold, where for every $x, y \in X$, the operator $x \square y$ on X is defined by $z \mapsto \{x, y, z\}$:

- (J₁) $\{x, y, z\}$ is symmetric bilinear in the outer variable x, z and conjugate linear in the inner variable y ,
- (J₂) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$ (Jordan triple identity),
- (J₃) $x \square x \in L(X, X)$ is a hermitian operator with spectrum ≥ 0 ,
- (J₄) $\|\{x, x, x\}\| = \|x\|^3$.

It is known [21, p. 523] that in this definition condition (J₄) can be replaced by $\|x \square x\| = \|x\|^2$ and that

$$\|x \square y\| \leq \|x\| \cdot \|y\|$$

holds for all $x, y \in X$. Then, we have

$$\|\{x, y, z\}\| \leq \|x\| \cdot \|y\| \cdot \|z\|, \quad \text{for all } x, y, z. \quad (2.1)$$

Example 2.1. Let S be a locally compact topological space and let $C_0(S)$ be the Banach space of all continuous complex valued functions f on S vanishing at infinity with $\|f\| = \sup |f(S)|$. Then $C_0(S)$ is a JB*-triple with $\{f, g, h\} = f \bar{g}h$.

A linear subspace $I \subset X$ is called a *subtriple* if $\{I, I, I\} \subset I$.

For every $a \in X$, let $Q_a : X \rightarrow X$ be the conjugate linear operator defined by $Q_a(x) = \{a, x, a\}$. This operator is called the quadratic representation and it satisfies the fundamental formula

$$Q_{Q_a(b)} = Q_a Q_b Q_a$$

for all $a, b \in X$. For every $x, y \in X$, the Bergman operator $B(x, y) \in L(X, X)$ is defined by

$$B(x, y) = I_X - 2x \square y + Q_x Q_y.$$

From (2.1), we have

$$\|B(x, y)\| \leq (1 + \|x\| \cdot \|y\|)^2, \quad x, y \in X. \quad (2.2)$$

In case $\|x \square y\| < 1$, the spectrum of $B(x, y)$ lies in $\{z \in \mathbb{C} : |z - 1| < 1\}$. In particular, the fractional power $B(x, y)^r \in GL(X)$ exists for every $r \in \mathbb{R}$ in a natural way (cf. [21, p. 517]).

Let B be the unit ball of a JB*-triple X . Then, for each $a \in B$, the Möbius transformation g_a defined by

$$g_a(x) = a + B(a, a)^{1/2} (I_X + x \square a)^{-1} x, \quad (2.3)$$

is a biholomorphic mapping of B onto itself with $g_a(0) = a$, $g_a(-a) = 0$ and $g_{-a} = g_a^{-1}$.

Proposition 2.2. Let g_a be as above. Then for any $a \in B$, g_a extends biholomorphically to a neighborhood of \bar{B} and we have

$$[Dg_a(0)]^{-1} D^2 g_a(0)(x, y) = -2\{x, a, y\}, \quad (2.4)$$

$$\|Dg_a(0)\| \leq 1, \quad (2.5)$$

$$\|[Dg_a(0)]^{-1}\| = \frac{1}{1 - \|a\|^2}, \quad (2.6)$$

$$Dg_{\zeta a}(0) = Dg_a(0), \quad |\zeta| = 1, \quad (2.7)$$

$$g_a(a) = \frac{2}{1 + \|a\|^2} a, \quad (2.8)$$

$$g_a(x) = x + a - \{x, a, x\} + O(\|a\|^2), \quad (2.9)$$

$$[Dg_a(0)]^{-1} = I_X + O(\|a\|^2). \quad (2.10)$$

Moreover, we have

$$\frac{1}{1 - \|g_{-z}(w)\|^2} \leq \frac{(1 + \|w\| \cdot \|z\|)^2}{(1 - \|w\|^2)(1 - \|z\|^2)}, \quad z, w \in B. \quad (2.11)$$

Proof. Since $\|x \square a\| \leq \|x\| \cdot \|a\|$, g_a and $g_a^{-1} = g_{-a}$ extend holomorphically to $\|x\| < 1/\|a\|$. Then, g_a extends biholomorphically to a neighborhood of \bar{B} . Since

$$g_a(x) = a + B(a, a)^{1/2}[x - (x \square a)x] + O(\|x\|^3) = a + B(a, a)^{1/2}[x - \{x, a, x\}] + O(\|x\|^3),$$

we have

$$Dg_a(x)(y) = B(a, a)^{1/2}[y - \{y, a, x\} - \{x, a, y\}] + O(\|x\|^2)$$

and

$$D^2 g_a(0)(y, z) = B(a, a)^{1/2}[-\{y, a, z\} - \{z, a, y\}] = -2B(a, a)^{1/2}\{y, a, z\}.$$

Since $Dg_a(0) = B(a, a)^{1/2}$, we obtain (2.4). By [22, Corollary 3.6], we obtain (2.5) and (2.6). Since

$$B(\zeta a, \zeta a) = B(a, a), \quad |\zeta| = 1,$$

we obtain (2.7). Since the JB^* -subtriple of X generated by a , denoted by X_a , is isometrically isomorphic to $C_0(S)$ for some locally compact subset $S \subset \mathbb{R}$ [21], it is easy to see that in X_a and hence in X , we have

$$g_a(a) = \frac{2}{1 + \|a\|^2} a.$$

Thus, we obtain (2.8). Since $B(a, a)^{1/2} = I_X + O(\|a\|^2)$, we have (2.10) and

$$g_a(x) = a + B(a, a)^{1/2}[x - \{x, a, x\}] + O(\|a\|^2) = a + x - \{x, a, x\} + O(\|a\|^2).$$

Since

$$\frac{1}{1 - \|g_{-z}(w)\|^2} = \|B(w, w)^{-1/2} B(w, z) B(z, z)^{-1/2}\|, \quad z, w \in B \quad (2.12)$$

by [25, Proposition 3.1], we obtain (2.11) from (2.2) and (2.6). \square

$x \in X$ is called *regular* if $x \square x \in GL(X)$ and $x \in X$ is called a *tripotent* if $\{x, x, x\} = x$. A point $u \in \bar{B}$ is said to be a *real* (resp. *complex*) *extreme point* of \bar{B} if the only $x \in X$ satisfying $\|u + \lambda x\| \leq 1$ for all real (resp. complex) numbers λ with $|\lambda| \leq 1$ is $x = 0$. We call $u \in \bar{B}$ *holomorphically extreme* in \bar{B} if for every open neighborhood U of $0 \in \mathbb{C}$ and every holomorphic mapping $f: U \rightarrow X$ the conditions $f(0) = u$ and $f(U) \subset \bar{B}$ imply that $f'(0) = 0$. $u \in \partial B$ is called a *simple boundary point* of B if $u + ty \in \partial B$, $y \in X$, $t \in \mathbb{C}$, $|t| < 1$ always implies $y = 0$. The following result is obtained in Kaup and Upmeyer [23, Proposition 3.5].

Proposition 2.3. Let B be the unit ball of a JB^* -triple X and $u \in X$. Then the following conditions are equivalent:

- (i) u is a regular tripotent in X ;
- (ii) u is holomorphically extreme in \bar{B} ;

- (iii) u is a complex extreme point of \bar{B} ;
- (iv) u is a simple boundary point of B .

Let \mathcal{E} be the set of all complex extreme points of \bar{B} . As a corollary of the above proposition, we obtain the following maximum principle for holomorphic mappings on the unit ball of a JB^* -triple. When B is the unit ball of a J^* -algebra, see Harris [18, Theorem 9]. By the Krein–Milman theorem (see e.g. [8, Chapter 4]), it is known that if \bar{B} is a compact subset of X , then \mathcal{E} is nonempty.

Proposition 2.4. *Let B be the unit ball of a JB^* -triple X and let \mathcal{E} denote the set of all complex extreme points of \bar{B} . If $\mathcal{E} \neq \emptyset$, then:*

- (i) *Let $g_a \in \text{Aut}(B)$ given in (2.3). Then $g_a(\mathcal{E}) = \mathcal{E}$ for any $a \in B$;*
- (ii) *Let Y be a complex Banach space. Let $f : B \rightarrow Y$ be a holomorphic mapping with a continuous and bounded extension to $B \cup \mathcal{E}$. Then*

$$\|f(x)\| \leq \sup\{\|f(u)\| : u \in \mathcal{E}\}, \quad x \in B.$$

Moreover, f is completely determined by its value on \mathcal{E} .

Proof. (i) Since $g_a^{-1} = g_{-a}$, it suffices to show that $g_a(\mathcal{E}) \subset \mathcal{E}$ for any $a \in B$. Let $v = g_a(u)$, where $u \in \mathcal{E}$. Assume that $v + \lambda x \in \bar{B}$ for $|\lambda| \leq 1$. Let

$$h(\lambda) = g_a^{-1}(v + \lambda x), \quad \lambda \in U.$$

Then h is holomorphic on U by Proposition 2.2, $h(0) = g_a^{-1}(v) = u$ and $h(U) \subset \bar{B}$. Since u is a holomorphic extreme point by Proposition 2.3, we must have $h'(0) = 0$. This implies that $Dg_a^{-1}(v)(x) = 0$. Since g_a^{-1} extends biholomorphically to a neighborhood of \bar{B} , we obtain $x = 0$. Thus, $v \in \mathcal{E}$.

(ii) By the mean value property for vector valued holomorphic functions, we obtain

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(g_x(e^{i\theta}u)) d\theta,$$

where $u \in \mathcal{E}$. Since $g_x(e^{i\theta}u) \in \mathcal{E}$ for $\theta \in [0, 2\pi]$ by (i), we obtain (ii). \square

3. Linear invariance in X

We define the notion of linear invariant families and the norm-order in the unit ball B of a complex Banach space X .

Definition 3.1. Let B be the unit ball of a complex Banach space X . Then a family \mathcal{F} is called a *linear-invariant family* if:

- (i) $\mathcal{F} \subset \mathcal{LS}(B)$,

and

- (ii) $\Lambda_\phi(f) \in \mathcal{F}$, for all $f \in \mathcal{F}$ and $\phi \in \text{Aut } B$,

where $\text{Aut } B$ denotes the set of biholomorphic automorphisms of B , and $\Lambda_\phi(f)$ is the *Koebe transform*

$$\Lambda_\phi(f)(x) = [D\phi(0)]^{-1} [Df(\phi(0))]^{-1} (f(\phi(x)) - f(\phi(0))), \quad (3.1)$$

for all $x \in B$.

Note that the Koebe transform has the group property $\Lambda_\psi \circ \Lambda_\phi = \Lambda_{\phi \circ \psi}$.

If \mathcal{F} is a linear invariant family, we define two types of *norm-order* of \mathcal{F} (cf. [29]), given by

$$\|\text{ord}\|_{X,1} \mathcal{F} = \sup_{f \in \mathcal{F}} \sup_{\|y\|=1} \left\{ \frac{1}{2} \|D^2 f(0)(y, \cdot)\| \right\}$$

and

$$\|\text{ord}\|_{X,2} \mathcal{F} = \sup_{f \in \mathcal{F}} \sup_{\|y\|=1} \left\{ \frac{1}{2} \|D^2 f(0)(y, y)\| \right\}.$$

It is clear that $\|\text{ord}\|_{X,1}\mathcal{F} \geq \|\text{ord}\|_{X,2}\mathcal{F}$. Since

$$D^2f(0)(y, z) = \frac{1}{2}\{D^2f(0)(y + z, y + z) - D^2f(0)(y, y) - D^2f(0)(z, z)\},$$

we obtain $\|\text{ord}\|_{X,1}\mathcal{F} \leq 3\|\text{ord}\|_{X,2}\mathcal{F}$. Moreover, if X is a Hilbert space, then $\|\text{ord}\|_{X,1}\mathcal{F} = \|\text{ord}\|_{X,2}\mathcal{F}$ by Hörmander [19, Theorem 4].

We now give some examples of linear invariant families in the unit ball B of a complex Banach space X .

Example 3.2. $S(B)$, the set of all biholomorphic mappings in $\mathcal{LS}(B)$. If X is a complex Hilbert space of dimension n , where $n > 1$, the linear invariant family $S(B)$ does not have finite norm-order (see [29], cf. [1]).

Example 3.3. $\mathcal{U}_\alpha(B)$, the union of all linear invariant families contained in $\mathcal{LS}(B)$ with norm-order not greater than α . This is a generalization of the universal linear invariant families $\mathcal{U}_\alpha = \mathcal{U}_\alpha(\Delta)$ considered in [30].

Example 3.4. If \mathcal{G} is a nonempty subset of $\mathcal{LS}(B)$, then the linear invariant family generated by \mathcal{G} is the family

$$\Lambda[\mathcal{G}] = \{\Lambda_\phi(g) : g \in \mathcal{G}, \phi \in \text{Aut } B\}.$$

The linear invariance is a consequence of the group property of the Koebe transform. Obviously, $\Lambda[\mathcal{G}] = \mathcal{G}$ if and only if \mathcal{G} is a linear-invariant family. In the case of the unit Euclidean ball and the unit polydisc in \mathbb{C}^n , this example provided a useful technique for generating many interesting mappings (see [26–28]). For example, we can use a single mapping f from $\mathcal{LS}(B)$ to generate the linear invariant family $\Lambda[\{f\}]$. The family $\Lambda[\{i\}]$, generated by the identity mapping $i(x) = x$, consists of all the Koebe transforms of $i(x)$.

Example 3.5. $\mathcal{K}(B)$, the set of convex mapping in $\mathcal{LS}(B)$.

As in the proof of [29, Theorem 5.1], we obtain the following result. We will see later that $\|\text{ord}\|_{X,2}\mathcal{K}(B) = 1$, if B is the unit ball of a JB^* -triple. We remark that if $X = \ell^1$ is the complex Banach space of summable complex sequences, then $\|\text{ord}\|_{X,2}\mathcal{K}(B) = 0$, since the only mapping $f \in \mathcal{K}(B)$ is the identity mapping [32, Corollary 1].

Proposition 3.6. Let B be the unit ball of a complex Banach space X and let $\mathcal{K}(B)$ be the set of normalized convex mappings on B . Then $\|\text{ord}\|_{X,2}\mathcal{K}(B) \leq 1$.

When B is the unit ball of a JB^* -triple X , we have the following first-order approximation formula for the Koebe transform of f .

Lemma 3.7. Let $g_a \in \text{Aut}(B)$ given in (2.3). If $f \in \mathcal{LS}(B)$, then

$$\begin{aligned} [Dg_a(0)]^{-1}[Df(g_a(0))]^{-1}(f(g_a(x)) - f(g_a(0))) \\ = f(x) + Df(x)(a - \{x, a, x\}) - a - D^2f(0)(a, f(x)) + O(\|a\|^2), \quad a \rightarrow 0. \end{aligned}$$

Proof. Since

$$f(x) = x + \frac{1}{2}D^2f(0)(x, x) + \frac{1}{6}D^3f(0)(x, x, x) + \cdots,$$

we have

$$f(g_a(0)) = a + O(\|a\|^2), \tag{3.2}$$

and

$$[Df(g_a(0))]^{-1} = I_X - D^2f(0)(a, \cdot) + O(\|a\|^2). \tag{3.3}$$

Since

$$f(x + y) = f(x) + Df(x)y + O(\|y\|^2),$$

we obtain from (2.9) that

$$f(g_a(x)) = f(x + a - \{x, a, x\} + O(\|a\|^2)) = f(x) + Df(x)(a - \{x, a, x\}) + O(\|a\|^2). \tag{3.4}$$

From (2.10) and (3.3), we have

$$[Dg_a(0)]^{-1}[Df(g_a(0))]^{-1} = I_X - D^2f(0)(a, \cdot) + O(\|a\|^2). \quad (3.5)$$

Then from (3.2), (3.4) and (3.5), we have

$$\begin{aligned} & [Dg_a(0)]^{-1}[Df(g_a(0))]^{-1}(f(g_a(x)) - f(g_a(0))) \\ &= (I_X - D^2f(0)(a, \cdot))[f(x) + Df(x)(a - \{x, a, x\}) - a] + O(\|a\|^2) \\ &= f(x) + Df(x)(a - \{x, a, x\}) - a - D^2f(0)(a, f(x)) + O(\|a\|^2). \end{aligned}$$

This completes the proof. \square

The following useful result is a natural extension to JB^* -triples of [30, Lemma 1.2] (cf. [12,13,27]).

Lemma 3.8. *Let \mathcal{F} be a linear-invariant family on the unit ball B of a JB^* -triple X with $\|\text{ord}\|_{X,1}\mathcal{F} = \alpha$ and $\|\text{ord}\|_{X,2}\mathcal{F} = \beta$. Then*

$$\alpha = \sup_{f \in \mathcal{F}} \sup \left\{ \left\| \frac{1}{2} \Phi(f, x, y, z) - \{y, x, z\} \right\| : \|x\| < 1, \|y\| = \|z\| = 1 \right\}, \quad (3.6)$$

and

$$\beta = \sup_{f \in \mathcal{F}} \sup \left\{ \left\| \frac{1}{2} \Phi(f, x, y, y) - \{y, x, y\} \right\| : \|x\| < 1, \|y\| = 1 \right\}, \quad (3.7)$$

where

$$\Phi(f, x, y, z) = [Dg_x(0)]^{-1}[Df(x)]^{-1}D^2f(x)(Dg_x(0)y, Dg_x(0)z).$$

Proof. It is clear that

$$\sup_{f \in \mathcal{F}} \sup \left\{ \left\| \frac{1}{2} \Phi(f, x, y, z) - \{y, x, z\} \right\| : \|x\| < 1, \|y\| = \|z\| = 1 \right\} \geq \alpha.$$

On the other hand, let $f \in \mathcal{F}$ and $\phi = g_x$ where $x \in B$. It is clear that $F \in \mathcal{F}$, where $F(w) = \Lambda_\phi(f)(w)$, $w \in B$. Therefore, we have

$$\left\| \frac{1}{2} D^2F(0)(y, z) \right\| \leq \alpha, \quad y, z \in X, \|y\| = \|z\| = 1. \quad (3.8)$$

If we differentiate twice the mapping $F = \Lambda_\phi(f)$, given by (3.1), we obtain that

$$DF(w) = [D\phi(0)]^{-1}[Df(\phi(0))]^{-1}Df(\phi(w))D\phi(w), \quad w \in B,$$

and

$$\begin{aligned} D^2F(w)(y, z) &= [D\phi(0)]^{-1}[Df(\phi(0))]^{-1}\{D^2f(\phi(w))(D\phi(w)y, D\phi(w)z) + Df(\phi(w))D^2\phi(w)(y, z)\}, \\ & \quad y, z \in X. \end{aligned}$$

Evaluating at $w = 0$, we obtain that

$$D^2F(0)(y, z) = \Phi(f, x, y, z) + [Dg_x(0)]^{-1}D^2g_x(0)(y, z).$$

Hence, from (2.4) and this equality, we have

$$D^2F(0)(y, z) = \Phi(f, x, y, z) - 2\{y, x, z\}.$$

Finally, from (3.8) and the last relation, one concludes that

$$\left\| \frac{1}{2} \Phi(f, x, y, z) - \{y, x, z\} \right\| \leq \alpha,$$

for all $x \in B$ and $y, z \in X$, $\|y\| = \|z\| = 1$. Thus, we obtain (3.6). Putting $z = y$ in the above argument, we obtain (3.7). This completes the proof. \square

Note that in the case of one complex variable, the relations (3.6) and (3.7) are equivalent to

$$\alpha = \beta = \sup_{f \in \mathcal{F}} \sup_{|b| < 1} \left| \frac{1}{2}(1 - |b|^2) \frac{f''(b)}{f'(b)} - \bar{b} \right|$$

(compare with [30, Lemma 1.2]).

Pfaltzgraff and Suffridge [29, Theorem 3.1] proved recently that if \mathcal{M} is a linear invariant family on the Euclidean unit ball of \mathbb{C}^n , then $\|\text{ord } \mathcal{M}\| \geq 1$. Hamada and Kohr obtained the extension of this result to the unit ball of a complex Hilbert space in [12, Theorem 3.2] and to the unit polydisc in [13, Theorem 3.2]. In the following we obtain the extension of this result to the unit ball of a JB^* -triple.

Theorem 3.9. *Let \mathcal{F} be a linear invariant family on the unit ball B of a JB^* -triple X . Then $\|\text{ord } \mathcal{F}\|_{X,2} \geq 1$.*

Proof. We will use an argument similar to that in the proof of [29, Theorem 3.1]. Let $\beta = \|\text{ord } \mathcal{F}\|_{X,2}$ and let $x \in B \setminus \{0\}$ be fixed. Putting $y = \frac{x}{\|x\|}$, $x \in B \setminus \{0\}$, in (3.7), we obtain that

$$\beta \geq \left\| \frac{1}{2\|x\|^2} \Phi(f, x, x, x) - \frac{1}{\|x\|^2} \{x, x, x\} \right\|,$$

where

$$\Phi(f, x, x, x) = [Dg_x(0)]^{-1} [Df(x)]^{-1} D^2 f(x) (Dg_x(0)x, Dg_x(0)x).$$

Therefore, we have

$$\beta \geq \left| \frac{1}{2\|x\|^2} z^* (\Phi(f, x, x, x) - 2\{x, x, x\}) \right|, \quad (3.9)$$

where $z^* \in T(\{x, x, x\})$. Further, let

$$h(\zeta) = \frac{\zeta}{2\|x\|^2} z^* (\Psi(f, \zeta, x)), \quad |\zeta| < \frac{1}{\|x\|},$$

where

$$\Psi(f, \zeta, x) = [Dg_x(0)]^{-1} [Df(\zeta x)]^{-1} D^2 f(\zeta x) (Dg_x(0)x, Dg_x(0)x).$$

Then h is a holomorphic function on $|\zeta| < 1/\|x\|$ and by (2.7)

$$\Phi(f, \zeta x, \zeta x, \zeta x) = \zeta^2 \Psi(f, \zeta, x), \quad |\zeta| = 1.$$

Since $h(0) = 0$, for every r with $r < 1/\|x\|$, there exists a value of ζ with $|\zeta| = r$ such that $\text{Re } h(\zeta) \leq 0$.

We now replace x by ζx and z^* by $\frac{\bar{\zeta}}{|\zeta|} z^*$ in (3.9), where $|\zeta| = 1$ so that $\text{Re } h(\zeta) \leq 0$. Then we deduce that

$$\beta \geq \left| h(\zeta) - \frac{\|x\|^3}{\|x\|^2} \right| \geq -\Re h(\zeta) + \|x\| \geq \|x\|,$$

because we have used the fact that $\text{Re } h(\zeta) \leq 0$. Hence, $\beta \geq \|x\|$ for all $x \in B$. Therefore $\|\text{ord } \mathcal{F}\|_{X,2} \geq 1$. This completes the proof. \square

As a corollary of Proposition 3.6 and Theorem 3.9, we obtain the following result (cf. [29, Theorem 5.1]).

Corollary 3.10. *Let B be the unit ball of a JB^* -triple X and let $\mathcal{K}(B)$ be the set of normalized convex mappings on B . Then $\|\text{ord } \mathcal{K}(B)\|_{X,2} = 1$.*

Next, we give a result on a lower bound for starlikeness. Hamada and Kohr [14] (cf. [17]) proved the following sufficient condition for starlikeness on the unit ball of a complex Banach space.

Proposition 3.11. *Let f be a locally biholomorphic mapping on the unit ball B of a complex Banach space with $f(0) = 0$. If*

$$\| [Df(x)]^{-1} D^2 f(x)(x, \cdot) \| \leq 1, \quad x \in B,$$

then f is a starlike mapping of order $1/2$ on B .

Using the above sufficient condition, we will prove the following theorem (cf. [29, Theorems 5.5 and 5.7]).

Theorem 3.12. Let \mathcal{F} be a linear-invariant family on the unit ball B of a JB^* -triple X with $\|\text{ord}\|_{X,1}\mathcal{F} = \alpha < \infty$. If $f \in \mathcal{F}$, then f is a starlike mapping of order $1/2$ on B_{r_s} , where $r_s \in (0, 1)$ is the unique solution of the equation

$$\frac{2r^2 + 2\alpha r}{(1 - r^2)^2} = 1.$$

Proof. From Lemma 3.8,

$$\| [Df(x)]^{-1} D^2 f(x)(Dg_x(0)y, Dg_x(0)z) \| \leq 2 \| Dg_x(0)\{y, x, z\} \| + 2\alpha \| Dg_x(0) \| \cdot \| y \| \cdot \| z \|.$$

Also, we have $\| Dg_x(0) \| \leq 1$ and $\| [Dg_x(0)]^{-1} \| \leq 1/(1 - \| x \|^2)$ from (2.5) and (2.6). Therefore, putting $y = [Dg_x(0)]^{-1}x$ and $z = [Dg_x(0)]^{-1}w$ with $\| w \| = 1$ and using (2.1), we obtain that

$$\begin{aligned} \| [Df(x)]^{-1} D^2 f(x)(x, w) \| &\leq 2 \| Dg_x(0) \| \cdot \| [Dg_x(0)]^{-1}x \| \cdot \| x \| \cdot \| [Dg_x(0)]^{-1}w \| \\ &\quad + 2\alpha \| Dg_x(0) \| \cdot \| [Dg_x(0)]^{-1}x \| \cdot \| [Dg_x(0)]^{-1}w \| \\ &\leq \frac{2r^2 + 2\alpha r}{(1 - r^2)^2}, \end{aligned}$$

where $r = \| x \|$. From Proposition 3.11, f is a starlike mapping of order $1/2$ on B_{r_s} . This completes the proof. \square

Before to give the following result, we have to introduce some notations, as follows. This result relates the radius of univalence of a linear invariant family with the radius of nonvanishing of this family.

Let

$$r_0 = r_0(\mathcal{F}) = \sup\{r > 0: f(x) \neq 0, 0 < \| x \| < r, f \in \mathcal{F}\}$$

and let $r_1 = r_1(\mathcal{F})$ denote the radius of univalence of the linear invariant family \mathcal{F} , i.e.

$$r_1 = \sup\{r > 0: f \text{ is univalent on } B_r, f \in \mathcal{F}\}.$$

Then, we obtain the following result. This result is a generalization of [30, Lemma 2.4], [29, Theorem 5.11], [12, Theorem 3.4] and [13, Theorem 3.5] to the unit ball of a JB^* -triple. We remark that if $\|\text{ord}\|_{X,1}\mathcal{F} = \alpha < \infty$, then $r_0 > 0$ from Theorem 3.12.

Theorem 3.13. Let \mathcal{F} be a linear invariant family on the unit ball B of a JB^* -triple X . Assume that $r_0(\mathcal{F}) > 0$. Then

$$r_1 = \frac{r_0}{1 + \sqrt{1 - r_0^2}}.$$

Proof. Let $f \in \mathcal{F}$ and $r \leq \frac{r_0}{1 + \sqrt{1 - r_0^2}}$. Also, let $y, z \in B_r$ with $y \neq z$. Let

$$F(w; x) = [Dg_x(0)]^{-1} [Df(g_x(0))]^{-1} (f(g_x(w)) - f(g_x(0))), \quad w, x \in B, \quad (3.10)$$

where g_x is the biholomorphic automorphism of B , given in (2.3). Clearly, $F(\cdot; x) \in \mathcal{F}$, for all $x \in B$, and if we set $x = y$ and $w = g_y^{-1}(z)$ in (3.10), we obtain that

$$F(g_y^{-1}(z); y) = [Dg_y(0)]^{-1} [Df(y)]^{-1} (f(z) - f(y)). \quad (3.11)$$

From (2.11), we obtain

$$1 - \| g_{-y}(z) \|^2 \geq \frac{(1 - \| y \|^2)(1 - \| z \|^2)}{(1 + \| y \| \cdot \| z \|^2)} > \frac{(1 - r^2)^2}{(1 + r^2)^2}.$$

Therefore, we have

$$\| g_y^{-1}(z) \| = \| g_{-y}(z) \| < \frac{2r}{1 + r^2} \leq r_0.$$

Since $g_y^{-1}(z) \neq 0$ for $y \neq z$, we have $F(g_y^{-1}(z); y) \neq 0$. Then, we conclude from (3.11) that $f(y) \neq f(z)$, that means f is univalent on B_r . Therefore, $r_1 \geq \frac{r_0}{1 + \sqrt{1 - r_0^2}}$. Also, since $r_0 > 0$, we deduce that $r_1 > 0$.

In the second part of this proof, we will show that $r_1 \leq \frac{r_0}{1+\sqrt{1-r_0^2}}$. To this end, let $x \in B$ with $0 < \|x\| < \frac{2r_1}{1+r_1^2}$. Then there exists $a \in B$ such that $x = g_a(a)$ and $0 < \|a\| < r_1$ by (2.8). After short computations, we obtain the following relations

$$F(a; a) = [Dg_a(0)]^{-1} [Df(a)]^{-1} (f(x) - f(a))$$

and

$$F(-a; a) = -[Dg_a(0)]^{-1} [Df(a)]^{-1} f(a),$$

where F is defined by (3.10). Therefore, we have

$$f(x) = Df(a)Dg_a(0)(F(a; a) - F(-a; a)).$$

Since $0 < \|a\| < r_1$, $F(a; a) \neq F(-a; a)$. Hence, $f(x) \neq 0$. This implies that $r_0 \geq \frac{2r_1}{1+r_1^2}$. This is equivalent to $r_1 \leq \frac{r_0}{1+\sqrt{1-r_0^2}}$. This completes the proof. \square

Corollary 3.14. Let \mathcal{F} be a linear invariant family on the unit ball B of a JB^* -triple X . Assume that $r_0(\mathcal{F}) = 1$. Then \mathcal{F} is a family of normalized univalent mappings on B .

4. Finite dimensional case

In this section, we consider about the linear invariant families on the unit ball B of a finite dimensional JB^* -triple X .

When the dimension of X is finite, we obtain the normality of a linear invariant family with finite norm-order. This is a generalization of [29, Theorem 3.2] to the unit ball of a finite dimensional JB^* -triple.

Theorem 4.1. Let \mathcal{F} be a linear invariant family on the unit ball B of a JB^* -triple X . If $\|\text{ord}\|_{X,2}\mathcal{F} = \beta < \infty$, then \mathcal{F} is a locally uniformly bounded family. In particular, in the case of a finite dimensional JB^* -triple X , \mathcal{F} is a normal family.

Proof. Since $\|\text{ord}\|_{X,1}\mathcal{F} \leq 3\beta < \infty$, we obtain from Theorem 3.12 that, if $f \in \mathcal{F}$, then the mapping \tilde{f} given by

$$\tilde{f}(x) = \frac{1}{r_s} f(r_s x)$$

is a normalized starlike mapping of order $1/2$ on B . By [17, Theorem 3.1] (see also [4, Theorem 2.2], [10, Corollary 14], [9, Theorem 3.1]), we obtain that

$$\|\tilde{f}(x)\| \leq \frac{\|x\|}{1 - \|x\|}, \quad x \in B.$$

Therefore, in view of Cauchy's integral formula, we obtain that there exist $\delta_1 > 0$ and $M_1 > 0$ which is independent of $f \in \mathcal{F}$, such that $\|f\|$ and $\|Df\|$ are bounded by M_1 on B_{δ_1} . Now apply this result to the Koebe transform $F(y; x) = \Lambda_{g_x}(f)(y)$, where $\|x\| \leq \delta_1$. If we set $y = x$, then we obtain that

$$\|f(g_x(x)) - f(x)\| = \|Df(x)Dg_x(0)F(x; x)\|$$

and

$$\|Df(g_x(x))\| = \|Df(x)Dg_x(0)DF(x; x)[Dg_x(x)]^{-1}\|.$$

Let $\delta_2 = 2\delta_1/(1 + \delta_1^2) > \delta_1$. Since for any $y \in B_{\delta_2}$, there exists $x \in B_{\delta_1}$ such that $y = g_x(x)$ by (2.8), we get a uniform bound M_2 for $\|f\|$ and $\|Df\|$ on the ball of radius δ_2 . We continue the above process and we obtain a sequence δ_j of the radius of the ball centered at zero on which we have a uniform bound M_j for $\|f\|$ and $\|Df\|$ over all $f \in \mathcal{F}$, where

$$\delta_{j+1} = \frac{2\delta_j}{1 + \delta_j^2}.$$

The sequence δ_j is increasing and bounded above by 1. Therefore, it has a limit. If the limit is r , then we have $r = 2r/(1+r^2)$. Therefore, $r = 1$. This implies that for any $R \in (0, 1)$, there exists a number j_0 such that $R < \delta_{j_0}$. Thus, f is uniformly bounded on B_R . This completes the proof. \square

Next, we will prove the distortion theorem. This theorem is a generalization of [29, Theorem 4.1] and [13, Theorem 4.2] to the unit ball of a finite dimensional JB^* -triple. For the distortion theorems of normalized convex mappings, see [2] and the references therein.

Theorem 4.2. Let \mathcal{F} be a linear invariant family on the unit ball B of a finite dimensional JB^* -triple X . If $\|\text{ord}\|_{X,1}\mathcal{F} = \alpha < \infty$, then

$$\|Df(x)\| \leq \frac{(1 + \|x\|)^{\alpha-1}}{(1 - \|x\|)^{\alpha+1}}, \quad x \in B$$

for all $f \in \mathcal{F}$.

As a corollary, we obtain the following growth theorem as in the proof of [29, Theorem 4.2]. The following theorem is a generalization of [29, Theorem 4.2] and [13, Theorem 4.3] to the unit ball of a finite dimensional JB^* -triple.

Theorem 4.3. Let \mathcal{F} be a linear invariant family on the unit ball B of a finite dimensional JB^* -triple X . If $\|\text{ord}\|_{X,1}\mathcal{F} = \alpha < \infty$, then

$$\|f(x)\| \leq \frac{1}{2\alpha} \left\{ \left(\frac{1 + \|x\|}{1 - \|x\|} \right)^\alpha - 1 \right\}$$

for all $f \in \mathcal{F}$ and $x \in B$.

We give a few remarks and lemmas in preparation for the proof of Theorem 4.2. Now, assume that \mathcal{F} is a linear invariant family which satisfies the assumptions of Theorem 4.2. Since \mathcal{F} is a normal family from Theorem 4.1, $\text{cl}(\mathcal{F})$ is also a linear invariant family such that $\|\text{ord}\|_{X,1}\text{cl}(\mathcal{F}) = \alpha$. Thus, we may assume that \mathcal{F} is compact in Theorems 4.2 and 4.3. For a point $x_0 = r_0 u_0$, where u_0 is a complex extreme point of \bar{B} and $0 < r_0 < 1$, we consider the extremal problem $\sup\{\|Dh(x_0)\| : h \in \mathcal{F}\}$, and let $F \in \mathcal{F}$ be an extremal mapping as follows:

$$\|DF(x_0)\| = \sup\{\|Dh(x_0)\| : h \in \mathcal{F}\}. \quad (4.1)$$

Then the definition of the operator norm insures the existence of a point y_0 with $\|y_0\| = 1$ such that $\|DF(x_0)\| = \|DF(x_0)y_0\|$. Clearly the extremal mapping also satisfies $\|DF(x_0)y_0\| \geq \|Dh(x_0)y_0\|$ for all $h \in \mathcal{F}$, and all points $y \in \bar{B}$.

Lemma 4.4. Let F be an extremal mapping defined by (4.1). Then it also has the extremal property

$$\|DF(x_0)\| = \max\{\|DF(e^{it}x_0)\| : 0 \leq t \leq 2\pi\}.$$

Proof. If this were false, then there would be a point $x_1 = e^{it}x_0$ with $0 < t < 2\pi$ such that $\|DF(x_1)\| > \|DF(x_0)\|$. Let $F_t(x) = e^{-it}F(e^{it}x)$ that must belong to the linear invariant family \mathcal{F} . Then $DF_t(x) = DF(e^{it}x)$ which yields the contradiction $\|DF_t(x_0)\| > \|DF(x_0)\|$. \square

We can prove the following *rotation lemma* by an argument similar to that in the proof of Pfaltzgraff and Suffridge [29, Lemma 4.4].

Lemma 4.5. Let F be an extremal mapping defined by (4.1). Then we obtain that $(DF(x_0)y_0)^*(D^2F(x_0)(x_0, y_0))$ is real.

Proof. For t near 0,

$$DF(e^{it}x_0)y_0 = DF(x_0 + (e^{it} - 1)x_0)y_0 = DF(x_0)y_0 + itD^2F(x_0)(x_0, y_0) + O(t^2).$$

Therefore,

$$0 \geq \text{Re}(DF(x_0)y_0)^*(DF(e^{it}x_0)y_0 - DF(x_0)y_0) = \text{Re}(DF(x_0)y_0)^*(itD^2F(x_0)(x_0, y_0)) + O(t^2).$$

Upon dividing by $|t|$ and considering both $t \rightarrow 0+$ and $t \rightarrow 0-$, we obtain that

$$\text{Im}(DF(x_0)y_0)^*(D^2F(x_0)(x_0, y_0)) = 0.$$

This completes the proof. \square

For r near r_0 we have the following formula.

Lemma 4.6. Let F be an extremal mapping defined by (4.1). Then we obtain that

$$\begin{aligned} \frac{\text{Re}(DF(x_0)y_0)^*(DF(ru_0)y_0 - DF(x_0)y_0)}{r - r_0} &= \frac{1}{1 - r_0^2} \{ \text{Re}(DF(x_0)y_0)^*(D^2F(0)(u_0, DF(x_0)(y_0))) \} \\ &\quad + \frac{r_0}{1 - r_0^2} \text{Re}(DF(x_0)y_0)^*(DF(x_0)[\{y_0, u_0, u_0\} + \{u_0, u_0, y_0\}]) \\ &\quad + O(r - r_0), \quad r \rightarrow r_0. \end{aligned}$$

Proof. Let $z = \alpha x_0 / \|x_0\|^2$, where $\alpha \in \mathbb{C} \setminus \{0\}$. For small $\|z\|$, the Koebe transform $F(x; z)$ can be viewed as a perturbation of the extremal $F(x)$. By Lemma 3.7, we have

$$\begin{aligned} F(x; z) &= Dg_z(0)^{-1} DF(g_z(0))^{-1} [F(g_z(x)) - F(g_z(0))] \\ &= F(x) + DF(x)(z - \{x, z, x\}) - z - D^2F(0)(z, F(x)) + O(\|z\|^2). \end{aligned}$$

Then we have

$$\begin{aligned} DF(x; z)(\cdot) &= DF(x)(\cdot) + D^2F(x)(z - \{x, z, x\}, \cdot) \\ &\quad - DF(x)\{\cdot, z, x\} - DF(x)\{x, z, \cdot\} - D^2F(0)(z, DF(x)(\cdot)) + O(\|z\|^2). \end{aligned}$$

Evaluating at x_0 and y_0 , we find that

$$\begin{aligned} &\operatorname{Re}(DF(x_0)y_0)^*(DF(x_0; z)(y_0) - DF(x_0)(y_0)) \\ &= \operatorname{Re}\{(DF(x_0)y_0)^*(D^2F(x_0)(z, y_0))\} - \operatorname{Re}\{(DF(x_0)y_0)^*(D^2F(x_0)(\{x_0, z, x_0\}, y_0))\} \\ &\quad - \operatorname{Re}\{(DF(x_0)y_0)^*(DF(x_0)[\{y_0, z, x_0\} + \{x_0, z, y_0\}])\} - \operatorname{Re}\{(DF(x_0)y_0)^*(D^2F(0)(z, DF(x_0)(y_0)))\} \\ &\quad + O(\|z\|^2). \end{aligned} \tag{4.2}$$

Since $\{u_0, u_0, u_0\} = u_0$ by Proposition 2.3, we have $\{x_0, z, x_0\} = \bar{\alpha}x_0$. Therefore, we obtain

$$\begin{aligned} \operatorname{Re}\{(DF(x_0)y_0)^*(D^2F(x_0)(\{x_0, z, x_0\}, y_0))\} &= \operatorname{Re}\{\bar{\alpha}(DF(x_0)y_0)^*(D^2F(x_0)(x_0, y_0))\} \\ &= \operatorname{Re}\{\alpha(DF(x_0)y_0)^*(D^2F(x_0)(x_0, y_0))\}, \end{aligned}$$

where the last equality follows from Lemma 4.5. We will divide (4.2) by $\|z\|$ and deduce a necessary condition as it tends to zero. Since

$$\operatorname{Re}(DF(x_0)y_0)^*(DF(x_0; z)(y_0) - DF(x_0)(y_0)) \leq 0,$$

we obtain

$$\begin{aligned} 0 &\leq -(1 - r_0^2) \operatorname{Re}\left\{(DF(x_0)y_0)^*\left(D^2F(x_0)\left(\frac{z}{\|z\|}, y_0\right)\right)\right\} \\ &\quad + \operatorname{Re}\left\{(DF(x_0)y_0)^*\left(DF(x_0)\left[\left\{y_0, \frac{z}{\|z\|}, x_0\right\} + \left\{x_0, \frac{z}{\|z\|}, y_0\right\}\right]\right)\right\} \\ &\quad + \operatorname{Re}\left\{(DF(x_0)y_0)^*\left(D^2F(0)\left(\frac{z}{\|z\|}, DF(x_0)(y_0)\right)\right)\right\} + O(\|z\|). \end{aligned}$$

We let $\|z\| \rightarrow 0$, noting that z can be replaced by $e^{it}z$ with arbitrary real t . Then we conclude that

$$\begin{aligned} 0 &= -(1 - r_0^2) \operatorname{Re}(DF(x_0)y_0)^*(D^2F(x_0)(u_0, y_0)) \\ &\quad + \operatorname{Re}(DF(x_0)y_0)^*(DF(x_0)[\{y_0, u_0, x_0\} + \{x_0, u_0, y_0\}]) \\ &\quad + \operatorname{Re}(DF(x_0)y_0)^*(D^2F(0)(u_0, DF(x_0)(y_0))), \end{aligned}$$

where $u_0 = x_0/r_0$, $r_0 = \|x_0\|$. Then we have

$$\begin{aligned} (1 - r_0^2) \operatorname{Re}(DF(x_0)y_0)^*(D^2F(x_0)(u_0, y_0)) &= r_0 \operatorname{Re}(DF(x_0)y_0)^*(DF(x_0)[\{y_0, u_0, u_0\} + \{u_0, u_0, y_0\}]) \\ &\quad + \operatorname{Re}(DF(x_0)y_0)^*(D^2F(0)(u_0, DF(x_0)(y_0))). \end{aligned}$$

By the usual expansion computation for r near r_0 and Lemma 4.5, we have the formula

$$\frac{\operatorname{Re}(DF(x_0)y_0)^*(DF(ru_0)y_0 - DF(x_0)y_0)}{r - r_0} = \operatorname{Re}(DF(x_0)y_0)^*(D^2F(x_0)(u_0, y_0)) + O(r - r_0).$$

Then the result follows by applying the preceding step. \square

Proof of Theorem 4.2. Since the dimension of X is finite, $\mathcal{E} \neq \emptyset$. It suffices to prove that for fixed $u_0 \in \mathcal{E}$,

$$\|Df(ru_0)\| \leq \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}} \tag{4.3}$$

for all $r \in (0, 1)$.

Throughout the proof, F will be an extremal mapping for the maximum problem associated with the points $x_0 = r_0 u_0$ and y_0 , i.e., $\|DF(x_0)y_0\| = \|DF(x_0)\| \geq \|Dh(x_0)\|$ for all $h \in \mathcal{F}$. We now assume that (4.3) is false. Hence there exist $r_0 \in (0, 1)$, a point y_0 with $\|y_0\| = 1$ and $F \in \mathcal{F}$ such that

$$\sup_{h \in \mathcal{F}} \|Dh(x_0)\| = \|DF(x_0)\| = \|DF(x_0)y_0\| > \frac{(1+r_0)^{\alpha-1}}{(1-r_0)^{\alpha+1}}.$$

Since the function

$$\psi(t, s) = \frac{(1+t)^{s-1}}{(1-t)^{s+1}} = \left(\frac{1+t}{1-t}\right)^s \frac{1}{1-t^2}, \quad 0 \leq t < 1,$$

is an increasing function of s such that $\lim_{s \rightarrow \infty} \psi(t, s) = \infty$ for $0 < t < 1$, there exists a number $\beta_0 > \alpha$ such that $\|DF(x_0)y_0\| = \psi(r_0, \beta_0)$. Consequently, for small $\varepsilon > 0$ and $r_0 - \varepsilon < r < r_0 + \varepsilon$ there is a C^1 smooth function $\beta(r)$ such that $\beta(r_0) = \beta_0$ and

$$\operatorname{Re}(DF(x_0)y_0)^*(DF(ru_0)y_0) = \frac{(1+r)^{\beta(r)-1}}{(1-r)^{\beta(r)+1}} = \psi(r, \beta(r)).$$

Thus

$$\frac{\operatorname{Re}(DF(x_0)y_0)^*(DF(ru_0)y_0 - DF(x_0)y_0)}{r - r_0} = \frac{\psi(r, \beta(r)) - \psi(r_0, \beta_0)}{r - r_0}.$$

From Lemma 4.6 and this equality, we have

$$\begin{aligned} \frac{\psi(r, \beta(r)) - \psi(r_0, \beta_0)}{r - r_0} &= \frac{1}{1-r_0^2} \operatorname{Re}(DF(x_0)y_0)^*(D^2F(0)(u_0, DF(x_0)(y_0))) \\ &\quad + \frac{r_0}{1-r_0^2} \operatorname{Re}(DF(x_0)y_0)^*(DF(x_0)[\{y_0, u_0, u_0\} + \{u_0, u_0, y_0\}]) \\ &\quad + O(r - r_0), \quad r \rightarrow r_0. \end{aligned}$$

As $r \rightarrow r_0$, the limit of the left-hand side of this formula is

$$(\psi_r + \beta' \psi_\beta)_{r=r_0} = \psi(r_0, \beta_0) \left\{ \frac{2(\beta_0 + r_0)}{1-r_0^2} + \beta'(r_0) \log \left(\frac{1+r_0}{1-r_0} \right) \right\}.$$

Noting that $\psi(r_0, \beta_0) = \|DF(x_0)\|$, we deduce that

$$\begin{aligned} \|DF(x_0)\| \left\{ \frac{2(\beta_0 + r_0)}{1-r_0^2} + \beta'(r_0) \log \left(\frac{1+r_0}{1-r_0} \right) \right\} &= \frac{1}{1-r_0^2} \operatorname{Re}(DF(x_0)y_0)^*(D^2F(0)(u_0, DF(x_0)(y_0))) \\ &\quad + \frac{r_0}{1-r_0^2} \operatorname{Re}(DF(x_0)y_0)^*(DF(x_0)[\{y_0, u_0, u_0\} + \{u_0, u_0, y_0\}]) \\ &\leq \|DF(x_0)\| \frac{2(r_0 + \alpha)}{1-r_0^2}. \end{aligned}$$

Our assumption that $\beta_0 > \alpha$ now implies that $\beta'(r_0) < 0$. Hence there exists an r with $0 < r < r_0$ such that $\beta(r) > \beta(r_0) = \beta_0 > \alpha$ and that

$$\operatorname{Re}(DF(x_0)y_0)^*(DF(ru_0)y_0) = \frac{(1+r)^{\beta(r)-1}}{(1-r)^{\beta(r)+1}} > \frac{(1+r)^{\beta_0-1}}{(1-r)^{\beta_0+1}}.$$

Let F_1 be an extremal such that

$$\|DF_1(ru_0)\| = \sup\{\|Dh(ru_0)\| : h \in \mathcal{F}\}.$$

We then have

$$\|DF_1(ru_0)\| \geq \frac{(1+r)^{\beta(r)-1}}{(1-r)^{\beta(r)+1}} > \frac{(1+r)^{\beta_0-1}}{(1-r)^{\beta_0+1}}.$$

Let

$$r_\infty = \inf \left\{ 0 < r < r_0 : \sup_{h \in \mathcal{F}} \|Dh(ru_0)\| > \frac{(1+r)^{\beta_0-1}}{(1-r)^{\beta_0+1}} \right\}.$$

Then there are a sequence $\{y_\nu\}$ with $\|y_\nu\| = 1$, a sequence of numbers $\{r_\nu\}$ decreasing to r_∞ and a sequence $\{F_\nu\} \subset \mathcal{F}$ such that

$$\|DF_\nu(x_\nu)\| = \|DF_\nu(x_\nu)y_\nu\| > \frac{(1+r_\nu)^{\beta_0-1}}{(1-r_\nu)^{\beta_0+1}}, \quad (4.4)$$

where $x_\nu = r_\nu u_0$. We may assume that y_ν tends to y with $\|y\| = 1$. Since \mathcal{F} is compact, there exists a subsequence of $\{F_\nu\}$ that converges locally uniformly to a mapping $f \in \mathcal{F}$. If $r_\infty > 0$, then

$$\|Df(r_\infty u_0)y\| \geq \frac{(1+r_\infty)^{\beta_0-1}}{(1-r_\infty)^{\beta_0+1}}.$$

Hence we can repeat the argument just given and deduce that there exists an r with $0 < r < r_\infty$ such that $\beta(r) > \beta_0 > \alpha$ and

$$\sup_{h \in \mathcal{F}} \|Dh(ru_0)\| = \frac{(1+r)^{\beta(r)-1}}{(1-r)^{\beta(r)+1}} > \frac{(1+r)^{\beta_0-1}}{(1-r)^{\beta_0+1}}.$$

This contradicts with the definition of r_∞ . So, $r_\infty = 0$.

From (4.4) we have

$$\|DF_\nu(x_\nu)y_\nu\| > 1 + 2\beta_0 r_\nu + O(r_\nu^2).$$

Equivalently one has the inequality

$$\begin{aligned} \|D^2 F_\nu(0)(r_\nu u_0, y_\nu) + O(r_\nu^2)\| &= \|DF_\nu(x_\nu)y_\nu - y_\nu\| \\ &> 1 + 2\beta_0 r_\nu - \|y_\nu\| + O(r_\nu^2) \\ &\geq 2\beta_0 r_\nu + O(r_\nu^2). \end{aligned}$$

Thus dividing by r_ν and letting $\nu \rightarrow \infty$ through the subsequence we obtain

$$\|D^2 f(0)(u_0, y)\| \geq 2\beta_0 > 2\alpha = 2\|\text{ord}\|_{X,1}\mathcal{F}.$$

This contradiction completes the proof. \square

Using Theorem 4.3 and arguments similar to those in the proof of [2, Lemma 2.7], we obtain the following result.

Theorem 4.7. Let \mathcal{F} be a linear invariant family on the unit ball B of a finite dimensional JB^* -triple X . If $\|\text{ord}\|_{X,1}\mathcal{F} = \alpha < \infty$, then for $x, y \in B$, we have

$$\|f(x) - f(y)\| \leq \frac{1}{2\alpha} (\exp(2\alpha C_B(x, y)) - 1) \min\{\|Df(x)Dg_x(0)\|, \|Df(y)Dg_y(0)\|\}.$$

Proof. Fix $x, y \in B$ and define $F : B \rightarrow X$ by

$$F(z) = [D\phi(0)]^{-1} [Df(y)]^{-1} (f(\phi(z)) - f(y)) \quad (z \in B),$$

where $\phi = g_y$. Then $F \in \mathcal{F}$. In view of Theorem 4.3, we have

$$\|F(z)\| \leq \frac{1}{2\alpha} \left\{ \left(\frac{1 + \|z\|}{1 - \|z\|} \right)^\alpha - 1 \right\} = \frac{1}{2\alpha} \{ \exp(2\alpha C_B(z, 0)) - 1 \}$$

for all $z \in B$. It follows that, for $z = g_y^{-1}(x)$,

$$\begin{aligned} \|f(x) - f(y)\| &= \|Df(y)Dg_y(0)F(z)\| \\ &\leq \|Df(y)Dg_y(0)\| \cdot \|F(z)\| \\ &\leq \|Df(y)Dg_y(0)\| \cdot \frac{1}{2\alpha} \{ \exp(2\alpha C_B(z, 0)) - 1 \} \\ &= \|Df(y)Dg_y(0)\| \cdot \frac{1}{2\alpha} \{ \exp(2\alpha C_B(x, y)) - 1 \}. \end{aligned}$$

Changing the roles of x and y , one deduces the desired result. \square

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